

Non-Hermitian Hamiltonian versus $E = 0$ localized states

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Abstract

We analyze the zero energy solutions, of a two dimensional system which undergoes a non-radial symmetric, complex potential $V(r, \phi)$. By virtue of the coherent states concept, the localized states are constructed, and the consequences of the imaginary part of the potential are found both analytically and schematically.

1 Introduction

Since the early years of quantum mechanics the exact solvability of quantum mechanical models have attracted much attention. Some exactly solvable models have already become typical standard examples in quantum mechanical textbooks. However, it was believed that the reality of the spectra of the Hamiltonians, describing quantum mechanical models, is necessarily attributed to their Hermiticity. It was the non-Hermitian PT -symmetric Hamiltonians proposed by Bender and Boettcher [1] that relaxed the Hermiticity condition as a necessity for the reality of the spectrum [1–7]. Herein, P denotes the parity ($PxP = -x$) and the anti-linear operator T mimics the time reflection ($TiT = -i$). Recently, Mostafazadeh [8] has introduced a broader class of non-Hermitian pseudo-Hermitian Hamiltonians (a generalization of PT -symmetric, therefore). In these settings [8–19], a Hamiltonian H is pseudo-Hermitian if it obeys the similarity transformation: $\eta H \eta^{-1} = H^\dagger$ where η is a Hermitian invertible linear operator. On the other hand, the study of the $E = 0$ bound states have found many applications in various fields [20–24]. Long ago, Daboul and Nieto [25,26] have discovered that, an attractive radial power low potential, $V(r) \sim r^{-\nu}$ for $\nu < -2$ and $\nu > 2$, passes through the $E = 0$ normalizable solutions. More recently Makowski and Górska, established the classical correspondence localized states of a system with zero energy and a general form

of power low potentials [27]. In their work, it was shown that the classical trajectories of the particle precisely matched with the localized quantum states.

In this work we attempt to understand the consequences of adding an imaginary term to a potential whose zero energy level passes through the $E = 0$ normalizable bound state solution (namely $V(r) = -\Gamma r^{-4}$). We believe that this kind of study is necessary, to find relations, if any between the non-Hermitian quantum mechanics and the classical mechanics.

The organization of our paper is as follows: In Sec. (2) we give an analytical solution to the Schrödinger equation of a zero energy particle under our chosen complex potential. We continue in Sec. (3) by adapting a closed form of the localized states from literature and then we set up and plot the classically equivalent coherent states of the system. We conclude our paper with Sec. (4).

2 Analytic solution of the Schrödinger equation

Two dimensional Schrödinger equation for a zero energy particle under a complex, non-radial symmetric potential is given by

$$\left[-\frac{\hbar^2}{2m} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + V(r, \phi) \right] \psi(r, \phi) = 0 \quad (1)$$

where

$$V(r, \phi) = -\frac{\Gamma}{r^4} - \frac{\Lambda}{r^2} e^{i\phi} \quad (2)$$

and for our purpose, Γ and Λ are some non-negative real constants. Before we go further, finding out the symmetric properties of $V(r, \phi)$ and consequently the Hamiltonian of the system may give some connections between this potential and the well known \mathcal{PT} -symmetric type potentials which are studied in the literature.

Let us introduce an operator Θ which is defined as

$$\Theta : i \rightarrow -i, \phi \rightarrow 2\pi - \phi \quad (3)$$

in which $i = \sqrt{-1}$ and ϕ is the usual azimuthal angle. One can easily show that Θ is a non-Hermitian, invertible operator whose inverse is given by

$$\Theta^{-1} = \Theta \quad (4)$$

and it can be decomposed into the two other operators \mathcal{T} and Π such that

$$\Theta = \Pi \mathcal{T} \quad (5)$$

in which the definition of these operators are given by

$$\begin{aligned} \Pi & : \phi \rightarrow 2\pi - \phi \\ \mathcal{T} & : i \rightarrow -i. \end{aligned} \quad (6)$$

Π is Hermitian and invertible such that $\Pi^{-1} = \Pi = \Pi^\dagger$, and \mathcal{T} is the usual time reversal operator. It is remarkable to observe that the Hamiltonian of the particle under the potential (2) is $\Pi\mathcal{T}$ -symmetric which means

$$H = H^{\Pi\mathcal{T}}. \quad (7)$$

Also it is said that the $\Pi\mathcal{T}$ -symmetry of a Hamiltonian H is unbroken if all of the eigenfunctions of H are simultaneously eigenfunctions of $\Pi\mathcal{T}$. It is easy to show that if the $\Pi\mathcal{T}$ -symmetry of a Hamiltonian H is unbroken, then the spectrum of H is real (to see the proof, one may see [30]).

We come back to the equation (1) and as usual, we take

$$\psi(r, \phi) = R(r) \Phi(\phi) \quad (8)$$

and after making substitution, in Eq. (1) we get a set of two equations as

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{l^2}{r^2} + \frac{2m\Gamma}{\hbar^2 r^4} \right] R(r) = 0 \quad (9)$$

and

$$\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + \frac{2m\Lambda}{\hbar^2} e^{i\phi} \Phi(\phi) = -l^2 \Phi(\phi) \quad (10)$$

where l is a constant to be identified.

By introducing a new dimensionless variable ρ as

$$\rho = \frac{r}{a_o} \quad (11)$$

where a_o is a positive constant, the Eqs. (9) and (10) become

$$\left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) - \frac{l^2}{\rho^2} + \frac{\gamma^2}{\rho^4} \right] R(r) = 0 \quad (12)$$

and

$$\frac{d^2 \Phi(\phi)}{d\phi^2} + \lambda^2 e^{i\phi} \Phi(\phi) = -l^2 \Phi(\phi). \quad (13)$$

in which

$$\gamma^2 = \frac{2m\Gamma}{\hbar^2 a_o^2} \quad (14)$$

and

$$\lambda^2 = \frac{2m\Lambda}{\hbar^2}. \quad (15)$$

In the angular part of the Schrödinger equation we change the variable, and introduce

$$\chi = e^{i\phi} \quad (16)$$

hence the Eq. (13) reads

$$\chi^2 \frac{d^2 \Phi(\chi)}{d\chi^2} + \chi \frac{d\Phi(\chi)}{d\chi} - (l^2 + \lambda^2 \chi) \Phi(\chi) = 0. \quad (17)$$

This is the modified Bessel ODE, such that its complete solution is well known as

$$\begin{aligned} \Phi(\chi) &= C_1 I_{2l}(2\lambda\sqrt{\chi}) + C_2 K_{2l}(2\lambda\sqrt{\chi}) \\ &\text{or} \\ \Phi(\phi) &= C_1 I_{2l}(2\lambda e^{i\phi/2}) + C_2 K_{2l}(2\lambda e^{i\phi/2}) \end{aligned} \quad (18)$$

in which $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions.

This solution should satisfy the following boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi) \quad (19)$$

or equivalently

$$\begin{aligned} C_1 I_{2l}(2\lambda e^{i\phi/2}) + C_2 K_{2l}(2\lambda e^{i\phi/2}) &= \\ C_1 I_{2l}(2\lambda e^{i(\phi+2\pi)/2}) + C_2 K_{2l}(2\lambda e^{i(\phi+2\pi)/2}). \end{aligned} \quad (20)$$

Since $I_\nu(z)$ and $K_\nu(z)$ are two independent solutions of the modified Bessel ODE, a class of solution is possible when we put $C_2 = 0$. Therefore we get

$$I_{2l}(2\lambda e^{i\phi/2}) = I_{2l}(2\lambda e^{i(\phi+2\pi)/2}). \quad (21)$$

In accordance with the following property of the modified Bessel functions [28]

$$I_\nu(z e^{m\pi i}) = e^{m\pi\nu i} I_\nu(z) \quad (22)$$

where z is a complex variable, m is an integer and ν is a real number, one can choose

$$z = 2\lambda e^{i\phi/2} \quad (23)$$

and $m = 1$ in Eq. (22) to get

$$I_{2l}(z e^{i\pi}) = e^{i2l\pi} I_{2l}(z). \quad (24)$$

This equality is valid for all z and l in their domains, but if one considers l to be an integer, we will get

$$I_{2l}(z e^{i\pi}) = I_{2l}(z) \quad (25)$$

which is equivalent to Eq. (22). Therefore l is found to be an integer i.e.

$$l = 0, \pm 1, \pm 2, \dots \quad (26)$$

Of course as a different possibility, one can choose $C_1 = 0$ to find a different class of solution but the following property [28]

$$K_\nu(z e^{im\pi}) = e^{-m\pi\nu i} K_\nu(z) - \pi i \sin m\nu\pi \csc \nu\pi I_\nu(z) \quad (27)$$

it comes to our setting as

$$K_{2l}(z e^{i\pi}) = e^{-\pi 2li} K_{2l}(z) - \pi i \sin 2l\pi \csc \nu\pi I_{2l}(z) \quad (28)$$

which obviously does not admit any solution.

As a result, the solutions of the angular part of the Schrödinger equation can be written explicitly as

$$\begin{aligned} \Phi_l(\phi) &= C_{l\lambda} I_{2l}(2\lambda e^{i\phi/2}) \\ l &= 0, 1, 2, \dots \end{aligned} \quad (29)$$

where, since $I_{-n}(z) = I_n(z)$, we just consider l to be non negative, and $C_{l\lambda}$ are the normalization constants given by

$$C_{l\lambda} = \sqrt{\frac{1}{\int_0^{2\pi} |\Phi_l(\phi)|^2 d\phi}} = \sqrt{\frac{1}{\int_0^{2\pi} |I_{2l}(2\lambda e^{i\phi/2})|^2 d\phi}}. \quad (30)$$

One may note that l still can be interpreted as the angular quantum number, since

$$\begin{aligned} \langle \hat{L}_{\Phi_l} \rangle &= \langle \Phi_l | -i\hbar \frac{\partial}{\partial \phi} | \Phi_l \rangle = \\ &= \langle C_{l\lambda} I_{2l}(2\lambda e^{i\phi/2}) | -i\hbar \frac{\partial}{\partial \phi} | C_{l\lambda} I_{2l}(2\lambda e^{i\phi/2}) \rangle = \\ -i\hbar |C_{l\lambda}|^2 &< I_{2l}(2\lambda e^{i\phi/2}) | i \left[\lambda e^{i\phi/2} I_{2l+1}(2\lambda e^{i\phi/2}) + l I_{2l}(2\lambda e^{i\phi/2}) \right] \rangle = l\hbar. \end{aligned} \quad (31)$$

The radial part of the Schrödinger equation, can be considered as the Bessel ODE if one defines

$$\xi = \frac{1}{\rho} \quad (32)$$

and therefore the Eq. (12) becomes

$$\xi^2 \frac{d^2}{d\xi^2} R(\xi) + \xi \frac{d}{d\xi} R(\xi) + (\gamma^2 \xi^2 - l^2) R(\xi) = 0 \quad (33)$$

which admits a complete solution

$$R_{l\gamma}(\xi) = C_1 J_l(\gamma\xi) + C_2 Y_l(\gamma\xi). \quad (34)$$

A physical, normalizable solution which for $l > 1$ corresponds to the bound state is given by [25,26]

$$\begin{aligned} R_{l\gamma}(\xi) &= N_{l\gamma} J_l(\gamma\xi) \\ l &= 2, 3, \dots \end{aligned} \quad (35)$$

in which $N_{l\gamma}$ are normalization constants given by

$$N_{l\gamma} = \frac{2}{a_o\gamma} \sqrt{\frac{(l+1)!}{(l-2)!}}. \quad (36)$$

Now we are ready to write the complete solution of the Schrödinger equation (i.e. wave function), by using the Eqs. (29) and (35) as

$$\psi_{l,\gamma,\lambda}(r, \phi) = C_{l\lambda} \frac{2}{a_o\gamma} \sqrt{\frac{(l+1)!}{(l-2)!}} I_{2l} \left(2\lambda e^{i\phi/2} \right) J_l \left(\frac{\gamma a_o}{r} \right). \quad (37)$$

We notice that, with $l > 1$, the only complex part of $\psi_{l,\gamma,\lambda}(r, \phi)$ is the modified Bessel function. As a matter of fact, the effect of Θ (introduced in Eq. (3)) on $\psi_{l,\gamma,\lambda}(r, \phi)$ is equivalent to the effect of Θ on $I_{2l} \left(2\lambda e^{i\phi/2} \right)$. Therefore, by using the expansion form of the modified Bessel function, this equation can be written as

$$\psi_{l,\gamma,\lambda}(r, \phi) = C_{l\lambda} \frac{2}{a_o\gamma} \sqrt{\frac{(l+1)!}{(l-2)!}} J_l \left(\frac{\gamma a_o}{r} \right) \sum_{s=0}^{\infty} \frac{\lambda^{2(s+l)}}{\Gamma(s+2l+1) s!} e^{i\phi(s+l)} \quad (38)$$

in which, clearly this is invariant under Θ (i.e., $\Theta \psi_{l,\gamma,\lambda}(r, \phi) = \psi_{l,\gamma,\lambda}(r, \phi)$).

2.1 A realized approach to the problem

In this section we will consider the following radial symmetric real potentials

$$V_{\pm}(r) = -\frac{\Gamma}{r^4} \pm \frac{\Lambda}{r^2} \quad (39)$$

where Γ and Λ are some positive constants as before. One should notice that the negative branch of the above potential is same as the potential in Eq. (2) in an attractive form, and the positive branch of it is same but in a repulsive form. The Schrödinger equation (1), after the usual separation method and change of variable, with the potential (38) comes to a set of two separated equations as

$$\left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) - \frac{\tilde{l}^2}{\rho^2} + \frac{\gamma^2}{\rho^4} \right] R(\rho) = 0 \quad (40)$$

and

$$\frac{d^2 \Phi(\phi)}{d\phi^2} = -\tilde{l}^2 \Phi(\phi) \quad (41)$$

where

$$\tilde{l}^2 = l^2 \pm \lambda^2$$

in which the positive (negative) sign is related to the $+\frac{\Lambda}{r^2}$ ($-\frac{\Lambda}{r^2}$), and the other factors are defined as before. One can easily show that, the final solution of the

Schrödinger equation with the potentials (38) can be written as

$$\psi_{l,\gamma,\lambda}(r, \phi) = \frac{1}{\sqrt{2\pi}} \frac{2}{a_o \gamma} \sqrt{\frac{(\tilde{l}+1)!}{(\tilde{l}-2)!}} e^{il\phi} J_{\tilde{l}}\left(\frac{1}{r}\right) \quad (42)$$

in which \tilde{l} must be greater than 1 to have bound states. In the sequel we will use the closed forms of the infinite number of degenerate wave functions (i.e., these states have same energy equal to zero), presented in the Eq.s (37) and (41) to construct the classical correspondence localized states.

3 Zero energy localized states

In references [25-27] it was shown that the trajectory of a classical particle which experiences a real potential in the form of the Γ -part of the potential considered in this work (2) (i.e. $-\frac{\Gamma}{r^4}$) is given by

$$\begin{cases} x = \frac{a}{2}(1 + \cos(\phi - \phi_o)) \\ y = \frac{a}{2}\sin(\phi - \phi_o) \end{cases} \quad (43)$$

which represents a circle with radius $a/2$ (i.e., if one set $\phi_o = 0$, this becomes $(x - a/2)^2 + y^2 = (a/2)^2$) such that

$$a = \sqrt{2m\Gamma/L^2} \quad (44)$$

and L is the conserved angular momentum of the particle. We notice that, the classical correspondence localized state (see [27] and the references therein) of the potential (2) while $\Lambda \rightarrow 0$ must have probability peak in accordance with the classical trajectory (i.e., a circle with radius $a/2$ as implied by Eq. (42) [27]). Our aim in the following is to see the effect of the Λ -part of the potential (2) on the shape of the classical correspondence localized states. To this end, first we find the localized states of the original potential (2) in a closed analytical form, then we follow similarly but for the case when the potential is in the forms of Eq. (38).

An available method to derive the corresponding localized states by using the solutions of the Schrödinger equation given in the previous sections, is based on the concept of deformed oscillator algebras [27,29]. Therefore a suitable explicit form of the localized state over the infinite number of degenerate states (with $E = 0$) $\psi_{l\gamma\lambda}(r, \phi)$ reads (see [27,29] and the references therein)

$$\Psi_N = \frac{1}{\sqrt{2\pi} (1 + |\tau|^2)^{N/2}} \sum_{k=0}^N \binom{N}{k}^{1/2} \tau^k \psi_{k,\gamma,\lambda}(r, \phi) \quad (45)$$

in which $k = l - 2$ and $\tau = Ae^{i\theta_o}$, where A and θ_o are some real constants.

3.1 Results

The classical correspondence localized states of a particle undergoes the potentials (2) and (38), by choosing $N = 7$ may be written as

$$\Psi_7 = \begin{cases} \frac{1}{8\sqrt{\pi}} \sum_{k=0}^7 C_{(2+k)\lambda} \sqrt{\frac{(\frac{\tau}{k})(k+3)!}{k!}} I_{(4+2k)} \left(2\lambda e^{i\frac{\phi}{2}} \right) J_{(2+k)} \left(\frac{1}{r} \right) & V = V_c \\ \frac{1}{16\pi} \sum_{k=0}^7 \sqrt{\frac{(\frac{\tau}{k})(\bar{l}+1)!}{(\bar{l}-2)!}} e^{i\bar{l}\phi} J_{\bar{l}} \left(\frac{1}{r} \right) & V = V_{\pm} \end{cases} \quad (46)$$

in which γa_o and A are set to be one and for convenience V_c and V_{\pm} refer to the potentials (2) and (38) respectively. Within the Figs (1) to (7) some density plot of $|\Psi_7|^2$ are given in terms of different values of λ . In the Fig. (1) we plot $|\Psi_7|^2$ with $\lambda = 0$ (i.e., $V_c = V_{\pm} = -\frac{\Gamma}{r^4}$) and the classical trajectory of the particle (this figure was reported in the Ref.[27]). In the Figs (2) to (4), (a), (b) and (c) refer to the potentials V_- , V_+ and V_c , respectively. In the Figs (5) and (6), (a) refers to V_+ and (b) refers to V_c . Finally Fig. (7) refers to V_c .

3.2 Behavior of the complexified modified Bessel functions

In this section we give a short description of the behavior of the complexified modified Bessel functions $I_{2l}(z)$ to explain why with $V = V_c$ the plot of the probability density with large values of λ are very localized around $\phi = 0$. It is not difficult to see that, in the case of $V = V_c$ the Λ -part of the original potential (2) goes into the ϕ -part of the Schrödinger equation and therefore the entire effect of this term appears in the ϕ -part of the wave function. Therefore the contribution of $C_{l\lambda} I_{2l}(2\lambda e^{i\phi/2})$ in the final form of the wave function, instead of the usual ϕ -part in the wave functions (i.e., $\frac{1}{\sqrt{2\pi}} e^{il\phi}$) of the case of $V = V_{\pm}$ is the reason of the great localization about small ϕ . Let us write

$$I_{2l} \left(2\lambda e^{i\phi/2} \right) = I_{2l} \left(2\lambda \cos \left(\frac{\phi}{2} \right) + i2\lambda \sin \left(\frac{\phi}{2} \right) \right) \quad (47)$$

which shows that the square root of the real (imaginary) part of the potential directly goes through the real (imaginary) part of the argument of $I_{2l}(2\lambda e^{i\phi/2})$. To see the behavior of $I_{2l}(2\lambda e^{i\phi/2})$ in terms of the real (imaginary) part of its argument we rewrite the last equation as

$$I_{2l} \rightarrow I_{2l} \left(2\lambda_1 \cos \left(\frac{\phi}{2} \right) + i2\lambda_2 \sin \left(\frac{\phi}{2} \right) \right) \quad (48)$$

which in the limit of $\lambda_1 = \lambda_2 = \lambda$ turns out to be $I_{2l}(2\lambda e^{i\phi/2})$. Figures (8),(9) and (10) show that once λ_2 vanishes $|I_{2l}|^2$ does not change much, but once λ_1 becomes zero, $|I_{2l}|^2$ decreases strongly. Also once λ_1 takes a larger value (it does not matter what is the value of λ_2), $|I_{2l}|^2$ takes much higher value close to

$\phi = 0$ or 2π . We conclude therefore that, the imaginary part of the argument of I_{2l} (and imaginary part of the potential therefore) does not contribute much in the localization of the $|\Psi_7|^2$ around small ϕ . In contrast, the real part, and its ϕ -dependent part (i.e., $\cos \phi$) causes such a great localization.

4 Conclusion

In conclusion, we concentrate ourselves and wish to comment on the figures. Fig. (1) is our reference figure, i.e., the classical correspondence localized state when the Λ -part of the potential vanishes. Fig.s (2-4) clearly show that the effects of the Λ -part in the V_c and V_+ cause a higher localization while in the V_- we see a lower localization. It is remarkable to observe that, the magnitude of the effects (within these three cases) V_c is greater than others. Fig.s (5,6) show that as λ takes large value the radius of the localized state corresponding to the V_+ decreases while the ϕ distribution of the $|\Psi_7|^2$ does not change. For the case of potential V_c the radii of the localized states are fixed while the ϕ distribution of the probability density $|\Psi_7|^2$ is changed so that the particle seems to be localized more around $\phi = 0$. In the Fig. (7) we see the effect of the Λ -part in the V_c as a great localization about $\phi = 0$. Obviously the figures imply that in the case of $V = V_c$, the particle is localized around $\phi = 0$, which is a direct consequence of the Λ -part of the potential. We notice that, our approach to the problem is in a closed analytical form, where all numerical results are based on the analytical solutions. There is no need to comment that any other approach will definitely lack the advantages of an exact analytical solution.

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Figure caption

Figure (1): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\epsilon = 0$ (i.e. the $V(r, \phi)$ is real) and $\lambda = 0.0$. This is the localized state corresponding to the classical trajectory of a particle which experiences just the first term of the potential, i.e. $V(r, \phi) = -\Gamma/r^4$ and therefore it is the reference plot.

Figure (2): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 0.1$. Also (a), (b) and (c) are correspondence with $V(r, \phi) = V_-, V_+$ and V_c respectively.

Figure (3): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 0.5$. Also (a), (b) and (c) are correspondence with $V(r, \phi) = V_-, V_+$ and V_c respectively.

Figure (4): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 1$. Also (a), (b) and (c) are correspondence with $V(r, \phi) = V_-, V_+$ and V_c respectively.

Figure (5): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 5$. Also (a) and (b) are correspondence with $V(r, \phi) = V_+$ and V_c respectively.

Figure (6): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 10$. Also (a) and (b) are correspondence with $V(r, \phi) = V_+$ and V_c respectively.

Figure (7): A plot of probability density $|\Psi_N(r, \phi)|^2$ for $N = 7$ (i.e. $2 \leq l \leq 9$), $\theta_o = 0$, $A = 1$, $\lambda = 100$.

Figure (8): A plot of $\left| I_{2l} \left(2\lambda_1 \cos\left(\frac{\phi}{2}\right) + i2\lambda_2 \sin\left(\frac{\phi}{2}\right) \right) \right|^2$ in terms of ϕ , for some different values of λ_1 and λ_2 .

Figure (9): A plot of $\left| I_{2l} \left(2\lambda_1 \cos \left(\frac{\phi}{2} \right) + i2\lambda_2 \sin \left(\frac{\phi}{2} \right) \right) \right|^2$ in terms of ϕ , for some different values of λ_1 and λ_2 .

Figure (10): A plot of $\left| I_{2l} \left(2\lambda_1 \cos \left(\frac{\phi}{2} \right) + i2\lambda_2 \sin \left(\frac{\phi}{2} \right) \right) \right|^2$ in terms of ϕ , for some different values of λ_1 and λ_2 .

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